

Chiral Topological Insulator on Nambu 3-Algebraic Geometry

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Chiral topological insulator (AIII-class) with Landau level is constructed based on the Nambu 3-algebraic geometry. We clarify the geometric origin of the chiral symmetry of the AIII-class topological insulator in the context of non-commutative geometry of 4D quantum Hall effect. The many-body groundstate wavefunction is explicitly derived as a $(l, l, l-1)$ Laughline-Halperin type wavefunction with unique K -matrix structure. Fundamental excitation is identified with anyonic string-like object with fractional charge $1/(1+2(l-1)^2)$. The Hall effect of the chiral topological insulators turns out to be a color version of Hall effect, which exhibits a dual property of the Hall and spin-Hall effects.

INTRODUCTION

In the past decade, the topological insulators (TIs) have attracted great attentions. Recently, two groups independently applied non-commutative geometry (NCG) techniques to TIs [1, 2] and discussed the appearance of quantum Nambu geometry [3–5] in the context of TIs. Since quantum Nambu geometry is closely related to the geometry of M-theory[6–9], the appearance of quantum Nambu geometry in TIs is quite intriguing, however the two groups reached a contradictory conclusion about the Nambu 3-bracket description for TIs; The authors of [1] insist that 3-algebra consistently describes physics of the TI, while the authors of [2] advocated the 3-algebra is not appropriate because of “pathological” properties of the 3-algebra. Here arises the question: (i) Which statement is correct or is there any compromise between these two?

The chiral TI is a class of TI also known as AIII-class TI that lives in arbitrary odd dimensions with chiral symmetry[23] and without either of time-reversal or particle-hole symmetry [10]. The chiral TI can be realized in 3D space and may possibly be relevant with daily experiment in laboratory, and the lattice model of the chiral TI has been proposed in [11]. In [1], the projection density operator method was applied to derive excitation energy within the single mode approximation, however the calculation cannot completely be carried out due to the lack of knowledge of the explicit form of the ground-state. Then arises the second equation: (ii) How can we reasonably construct the explicit groundstate wavefunction of the chiral TI?

In Ref.[12], the author clarified relations between the A-class TIs and quantum Hall effect (QHE) in arbitrary even dimensions. A-class and AIII-class TIs share many

similar properties: Both A-class and AIII-class are classified by \mathbb{Z} topological number and regularly appear in even and odd dimensions, and either of them does not respect time-reversal or particle-hole symmetry. However there is one discrepancy: AIII-class respect the chiral symmetry while A-class does not. Since A-class TIs are realized as QHE in even dimensions, the AIII-class TIs may be regarded as odd dimensional analogue of QHE. If so, it is reasonable why A and AIII-class TIs are so much like. At the same time, the third question arises: (iii) Why does only AIII-class have the chiral symmetry? There are not many works about QHE in odd dimension except for a pioneering work of Nair and Randjbar-Daemi [13] where they found the Landau level spectrum depends on a “mysterious” extra parameter whose counterpart does not exist in the even dimensional case. Here arises the last question: (iv) What is the physical meaning of the extra parameter found in Nair and Randjbar-Daemi’s analysis?

In this paper, we explore 3D chiral TI with emphasis on its relation to quantum Nambu geometry. Through the work, we provide convincing resolutions to all of the controversial issues from (i) to (iv).

THE LANDAU PROBLEM ON S^3

We first revisit the $SO(4)$ Landau model on a three-sphere [13] in the $SU(2)$ monopole background:

$$H = \frac{1}{2M} \sum_{\mu < \nu = 1}^4 \Lambda_{\mu\nu}^2, \quad (1)$$

The covariant angular momentum is constructed as $\Lambda_{\mu\nu} = -ix_\mu D_\nu + ix_\nu D_\mu$ where the covariant derivative is given by $D_\mu = \partial_\mu + iA_\mu$ with $SU(2)$ monopole gauge field

$$A_\mu = (A_i, A_4) = \left(-\frac{1}{2(1+x_4)}\epsilon_{ijk}x_j\sigma_k, 0\right). \quad (2)$$

Here, $\frac{1}{2}\sigma_i$ denote the $SU(2)$ matrices with spin magnitude $I/2$. The corresponding field strength $F_{\mu\nu} = \partial_\mu A_\nu -$

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$\partial_\nu A_\mu + i[A_\mu, A_\nu]$ is given by $F_{ij} = -x_i A_j + x_j A_i + \frac{1}{2}\epsilon_{ijk}\sigma_k$, and $F_{i4} = (1 + x_4)A_i = -\frac{1}{2}\epsilon_{ijk}x_j\sigma_k$, which satisfy $\sum_{\mu<\nu} F_{\mu\nu}^2 = \frac{1}{4}\sigma_i^2 = \frac{1}{4}I(I+2)$ and $\sum_{\mu<\nu} F_{\mu\nu}\tilde{F}_{\mu\nu} = 0$, with $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$. The Hamiltonian (1) may respect the $SO(4)$ symmetry since the the $SU(2)$ monopole magnetic field is perpendicular to the surface of S^3 ($x_\mu F_{\mu\nu} = F_{\mu\nu}x_\nu = 0$). The $SO(4)$ total angular momentum is constructed as $L_{\mu\nu} = \Lambda_{\mu\nu} + F_{\mu\nu}$, which satisfy $[L_{\mu\nu}, O_{\rho\sigma}] = i\delta_{\mu\rho}O_{\nu\sigma} - i\delta_{\mu\sigma}O_{\nu\rho} + i\delta_{\nu\sigma}O_{\mu\rho} - i\delta_{\nu\rho}O_{\mu\sigma}$, where $O_{\mu\nu} = L_{\mu\nu}, \Lambda_{\mu\nu}, F_{\mu\nu}$. One may confirm that H is indeed invariant under the $SO(4)$ transformations, $[H, L_{\mu\nu}] = 0$. The $SO(4)$ algebra consists of two independent $SU(2)$ algebras, $SU(2)_L \oplus SU(2)_R$, as $L_i^\pm = \Lambda_i^\pm + F_i^\pm$, where $O_i^\pm \equiv \frac{1}{2}\sum_{\mu<\nu}\eta_{\mu\nu}^{\pm i}O_{\mu\nu}$, with $O = \Lambda, F, L$ and t'Hooft tensor $\eta_{\mu\nu}^{\pm i} = \epsilon_{\mu\nu i4} \pm \delta_{\mu i}\delta_{\nu 4} \mp \delta_{\mu 4}\delta_{\nu i}$. The $SO(4)$ Landau Hamiltonian can be decomposed to two $SU(2)$ invariant Hamiltonians:

$$H = H_L + H_R, \quad (3)$$

where $H_{L/R} = \frac{1}{M}\Lambda_i^{\pm 2}$. Due to the relations $F_i^\pm \Lambda_i^\pm = \Lambda_i^\pm F_i^\pm = 0$, they can be rewritten as $H_{L/R} = \frac{1}{M}(L_i^{\pm 2} - F_i^{\pm 2}) = \frac{1}{M}(L_i^{\pm 2} - \frac{1}{16}I(I+2))$, with $\Lambda_i^\pm = \mathcal{L}_i^\pm + \frac{1}{4}\left((1-x_4)\delta_{ik} - \frac{1}{1+x_4}x_ix_k \pm \frac{x_4}{1+x_4}\epsilon_{ijk}x_j\right)\sigma_k$, $\mathcal{L}_i^\pm = -i\sum_{\mu<\nu}\eta_{\mu\nu}^{\pm i}x_\mu\partial_\nu = -i\frac{1}{2}\epsilon_{ijk}x_j\partial_k \mp i\frac{1}{2}x_i\partial_4 \pm i\frac{1}{2}x_4\partial_i$, $F_i^\pm = \frac{1}{4}(x_4\delta_{ik} + \frac{1}{1+x_4}x_ix_k \pm \epsilon_{ijk}x_j)\sigma_k$, and so $L_i^\pm = \Lambda_i^\pm + F_i^\pm = \mathcal{L}_i^\pm + \frac{1}{4}(\delta_{ik} \mp \frac{1}{1+x_4}\epsilon_{ijk}x_j)\sigma_k$. L_i^+ and L_i^- are interchanged under the “parity” transformation: $(x_i, x_4) \rightarrow (-x_i, x_4)$. Notice that though $H^{L/R}$ take a superficially similar form of the $SO(3)$ Landau Hamiltonian on the Haldane’s sphere [14], they are $SU(2)$ matrix valued Hamiltonians. The eigenvalues are readily derived as $E_{l_L, l_R} = \frac{1}{M}\left(l_L(l_L+1) + l_R(l_R+1)\right) - \frac{1}{8M}I(I+2)$ where l_L and l_R denote the $SU(2)_L \otimes SU(2)_R$ angular momentum indices.

With a given monopole charge $I/2$, the eigenvalues of the $SU(2)_L$ and $SU(2)_R$ angular momentum indices are related as [13]

$$l_L + l_R = n + \frac{I}{2}, \quad l_L - l_R = -s. \quad (4)$$

Here n denotes the Landau level index ($n = 0, 1, 2, \dots$), and s corresponds to the extra parameter that takes integer of half-integer values[24]. Therefore, l_L and l_R can respectively be expressed as $l_L = \frac{1}{2}(n + \frac{I}{2} + s)$ and $l_R = \frac{1}{2}(n + \frac{I}{2} - s)$. Notice that under the sign change of s , L_+ and L_- are interchanged, and hence s can be identified with the chirality index. The energy eigenvalues are rewritten as

$$E_n^{(s)} = \frac{1}{2MR^2}(n(n+2) + \frac{I}{2}(2n+1) + s^2), \quad (5)$$

and the corresponding n th Landau level degeneracy is given by

$$d_n^{(s)} = (2l_L+1)(2l_R+1) = (n + \frac{I}{2} - s + 1)(n + \frac{I}{2} + s + 1). \quad (6)$$

In the thermodynamic limit $I, R \rightarrow \infty$ with fixed $B = I/(2R^2)$ and finite s , $E_n^{(s)}$ reproduces the ordinary Landau level on 2D-plane, $\frac{B}{M}(n + \frac{1}{2})$. Notice that both of the energy eigenvalue (5) and the degeneracy (6) depend on s , and exhibit the chiral symmetry with respect to $s \rightarrow -s$. In the lowest Landau level (LLL) $n = 0$, the energy is represented as

$$E_{\text{LLL}}^{(s)} = \frac{1}{2MR^2}s^2 + \frac{I}{4MR^2}, \quad (7)$$

where due to the constraints (4), s takes $0, \pm 1, \pm 2, \dots, \frac{I}{2} - 1, \frac{I}{2}$ for even I , while $\pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots, \pm(\frac{I}{2} - 1), \pm \frac{I}{2}$ for odd I . Therefore, the minimum energy of (7) is achieved at $s = 0$ for even I , and at $s = \pm 1/2$ for odd I . It should be emphasized that for odd I , the LLL has “two fold” degeneracy coming from $s = 1/2$ and $s = -1/2$:

$$D(I) = d_{\text{LLL}}^{(s=1/2)} + d_{\text{LLL}}^{(s=-1/2)} = \frac{1}{2}(I+1)(I+3). \quad (8)$$

THE CHIRAL HOPF MAP AND QUATERNIONS

Let us consider the LLL basis states for $s = \pm 1/2$. We derive their functional form instead of the abstract Wigner \mathcal{D} -function [13]. For this purpose, we first introduce the chiral Hopf map:

$$S_L^3 \otimes S_R^3 \xrightarrow{S_D^3} S^3. \quad (9)$$

The coordinates of $S_L^3 \otimes S_R^3$ are expressed by the two-component complex spinors ψ_L and ψ_R (chiral Hopf spinors) subject to the normalization condition, $\psi_L^\dagger \psi_L = \psi_R^\dagger \psi_R = \frac{1}{2}$, and the chiral Hopf map is explicitly realized as

$$\psi_L, \psi_R \longrightarrow x_\mu = \psi_R^\dagger q_\mu \psi_L + \psi_L^\dagger \bar{q}_\mu \psi_R, \quad (\mu = 1, 2, 3, 4) \quad (10)$$

where q_μ and \bar{q}_μ denote the quaternions and conjugate-quaternions, $q_\mu = (q_i, 1) = (-i\sigma_i, 1)$ and $\bar{q}_\mu = (-q_i, 1) = (i\sigma_i, 1)$ [25]. It is straightforward to show that x_μ (10) obey $\sum_{\mu=1}^4 x_\mu x_\mu = 4(\psi_L^\dagger \psi_L) \cdot (\psi_R^\dagger \psi_R) = 1$. x_μ is invariant under the simultaneous $SU(2)$ transformation of ψ_L and ψ_R : $\psi_{L/R} \rightarrow \psi_{L/R} e^{i\alpha_i q_i}$. The explicit form of the chiral Hopf spinors is given by

$$\psi_{L/R}(x) = \frac{1}{\sqrt{2}} M_{L/R}(x) \phi, \quad (11)$$

where ϕ denotes a normalized two-component spinor (S^3 -fibre) with the normalization $\phi^\dagger \phi = 1$ and $M_{L/R}$ is the

following unitary matrix:

$$M_{L/R}(x) = \frac{1}{\sqrt{2(1+x_4)}}(1 + x_\mu q_\mu^{R/L}), \quad (12)$$

with $q_\mu^L \equiv q_\mu$ and $q_\mu^R \equiv \bar{q}_\mu$, and M_R is a quaternionic conjugate of M_L : $M_R(x) = (M_L(x))^\dagger = (M_L(x))^{-1}$. Notice that ψ_L and ψ_R are related by the “parity transformation”: $\psi_R(x_i, x_4) = \psi_L(-x_i, x_4)$, which is equivalent to the quaternionic conjugate in (12). We can derive the $SU(2)$ gauge field (2) as $A = -iM_R dM_L - iM_L dM_R$. One may readily verify that M^\pm satisfies $L_i^+ M_L = \frac{1}{2} M_L \sigma_i$, $L_i^- M_R = \frac{1}{2} M_R \sigma_i$, $L_i^+ M_R = L_i^- M_L = 0$, and so ψ_L and ψ_R respectively transform as $SU(2)_L \otimes SU(2)_R$ Weyl spinors, $(1/2, 0)$ and $(0, 1/2)$. The direct product of the two Weyl spinors gives the $SU(2)_L \otimes SU(2)_R$ “bi-spin” representation (l_L, l_R) that corresponds to the LLL basis state for $(l_L, l_R) = (\frac{1}{2}(\frac{I}{2} - s), \frac{1}{2}(\frac{I}{2} + s))$. With use of the chiral Hopf spinors, the (l_L, l_R) representation is constructed as

$$\Psi_{l_L, m_L}^L \otimes \Psi_{l_R, m_R}^R, \quad (13)$$

where

$$\Psi_{l_L, m_L}^L = \frac{1}{\sqrt{(l_L + m_L)!(l_L - m_L)!}} (\psi_L^1)^{l_L + m_L} (\psi_L^2)^{l_L - m_L}, \quad (14)$$

with $m_L = -l_L, -l_L + 1, \dots, l_L - 1, l_L$. Same for Ψ_{l_R, m_R}^R by replacing L with R . Thus, the LLL basis states are given by the holomorphic function of ψ_L and ψ_R . The chiral Hopf map is naturally derived by the “dimensional reduction” of the 2nd Hopf map, $S^7 \xrightarrow{S^3} S^4$. The 2nd Hopf map is realized as a map from a four-component complex spinor ψ subject to $\psi^\dagger \psi = 1$ to $x_a = \psi^\dagger \gamma_a \psi$ ($a = 1, \dots, 5$) with the $SO(5)$ gamma matrices, $\{\gamma_\mu, \gamma_5\} = \left\{ \begin{pmatrix} 0 & \bar{q}_\mu \\ q_\mu & 0 \end{pmatrix}, \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \right\}$ [15], and the chiral Hopf map is obtained by imposing an additional constraint $\psi^\dagger \gamma_5 \psi = \psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R = 0$ with $\psi = (\psi_L, \psi_R)^t$. This implies a geometric embedding of the chiral TI in 4D QHE. Similarities between the chiral TI and 4D QHE can also be found in the $SU(2)$ -bundle topology. With use of the chiral Hopf spinors, Q matrix in Ref.[10] is derived as $Q = 1 - 2\psi_L \psi_L^\dagger - 2\psi_R \psi_R^\dagger = \begin{pmatrix} 0 & g \\ g^\dagger & 0 \end{pmatrix}$, with $g = -x_\mu \bar{q}_\mu$, and the corresponding winding number is evaluated as $c_2 = \frac{(-i)}{3!2A(S^3)} \int_{S^3} \text{tr}(-ig^\dagger dg)^3 = \frac{1}{6} I(I+1)(I+2)$, which is exactly equivalent to the Chern number of the $SU(2)$ monopole-bundle over S^4 – the set-up of 4D QHE.

QUANTUM NAMBU GEOMETRY

Since the LLL basis states are given by the holomorphic function of the chiral Hopf spinors, the complex conjugation can be regarded as the derivative, $\psi_{L,R}^* \rightarrow$

$\partial/\partial\psi_{L,R}$. From the chiral Hopf map, we obtain the effective operator expression for the S^3 coordinates:

$$X_\mu = \alpha \psi_R^t q_\mu \frac{\partial}{\partial \psi_L} + \alpha \psi_L^t \bar{q}_\mu \frac{\partial}{\partial \psi_R}, \quad (15)$$

where $\alpha = 2R/I$. From the algebras of quaternions[26], we have

$$[X_\mu, X_\nu] = 2i\alpha(\eta_{\mu\nu}^i X_i^+ + \bar{\eta}_{\mu\nu}^i X_i^-), \quad (16)$$

where $X_i^+ = \alpha \psi_L^t \sigma_i \frac{\partial}{\partial \psi_L}$ and $X_i^- = \alpha \psi_R^t \sigma_i \frac{\partial}{\partial \psi_R}$ are two independent $SU(2)$ operators. Eq.(16) realizes the chiral symmetric version of the NC algebra of 4D QHE[15]. We also have $[X_\mu, X_i^\pm] = -i\alpha \eta_{\mu\nu}^\pm X_\nu^\pm$. In total, the ten operators, X_μ , X_i^+ and X_i^- , amount to form the $SO(5)$ algebra. The parameter s (4) denotes the eigenvalue of the chiral charge operator:

$$Q_5 \equiv \frac{1}{2} \psi_R^t \frac{\partial}{\partial \psi_R} - \frac{1}{2} \psi_L^t \frac{\partial}{\partial \psi_L}. \quad (17)$$

Meanwhile in the set-up of 4D QHE, the 5th coordinate of S_F^4 is given by $X_5 = \alpha \psi^t \gamma_5 \frac{\partial}{\partial \psi} = \alpha(\psi_L^t \frac{\partial}{\partial \psi_L} - \psi_R^t \frac{\partial}{\partial \psi_R})$, and then Q_5 corresponds to $Q_5 = -\frac{1}{2\alpha} X_5$. Remember in the set-up of 3D QHE, s was just an internal parameter, but in the “virtual” 4D QHE, s can be interpreted as the latitude of S^4 . The chiral symmetry realizes as the reflection symmetry of the 4D QHE with respect to the equator. This is the resolution for (iv).

In the precedent studies of NCG [16, 17], more elegant formulation of S_F^3 based on the quantum Nambu bracket is known [5, 6, 17]. One may readily confirm that matrix realization of X_μ (15) indeed satisfy the quantum Nambu-algebra for S_F^3 :

$$[X_\mu, X_\nu, X_\rho]_\chi = 2(I+2)\alpha^2 \epsilon_{\mu\nu\rho\sigma} X_\sigma, \quad (18)$$

where the chiral 3-bracket is defined as $[X_\mu, X_\nu, X_\rho]_\chi \equiv [X_\mu, X_\nu, X_\rho, Q_5]$. Under the definition of the ordinary quantum Nambu 3-bracket, $[X_\mu, X_\nu, X_\rho] \equiv [X_\mu X_\nu X_\rho]$, X_μ (15) do not form a closed algebra. Here, several comments are added. Firstly, the chiral 3-bracket can evade the pathological property of the 3-bracket emphasized in [2] as found $[X_\mu, X_\nu, 1]_\chi = 0$. Though the ordinary definition of the 3-bracket was adopted in Ref.[1], the whole 3-bracket algebra was not really used in the analysis, and hence the pathological property of the 3-bracket did not apparently appear. This gives the resolution for (i). Secondly, the chiral 3-bracket is well defined by being embedded in the four-bracket algebra of 4D QHE [12]. This is the algebraic evidence that the chiral TI naturally realizes as a “subspace” of the 4D QHE. Thirdly, the right-hand side of (18) suggests the existence of 3-rank $U(1)$ magnetic field $G_{\mu\nu\rho} = \frac{1}{r^4} \epsilon_{\mu\nu\rho\sigma} x_\sigma$. The corresponding gauge field is given by a 2-rank antisymmetric tensor field $C_{\mu\nu} = -C_{\nu\mu}$ ($G_{\mu\nu\rho} = \partial_\mu C_{\nu\rho} + \partial_\nu C_{\rho\mu} + \partial_\rho C_{\mu\nu}$) that couples to string-like object. Such 2-rank tensor field is simply obtained by the dimensional reduction of the 3-rank

tensor gauge field C_{abc} in the 4D QHE by $C_{\mu\nu} \equiv C_{\mu\nu 5}$, and so the string-like object from membrane excitation in 4D QHE.

The chiral Nambu 3-algebra gives a crucial implication for the existence of the chiral symmetry. The precedent studies [16, 17] tell that the fuzzy three-sphere is realized as a composite of two latitudes $s = 1/2$ and $s = -1/2$ of fuzzy four-sphere not just as the equator ($s = 0$). The reason is simple [17]; If the fuzzy three-sphere was simply the equator, the NC algebra would vanish, $[X_\mu, X_\nu, X_\rho]_\chi = [X_\mu, X_\nu, X_\rho, Q_5] = 0$ ($s = 0$). To incorporate a non-trivial NC structure, the fuzzy three-sphere has to be a composite of two S^3 s with opposite latitudes of same magnitude [17]. This suggests that, in the language of TIs, the chiral TI is given by a superposition of two S^3 s with opposite chiral charges on the virtual 4D QHE [Fig.1]. In other words, the requirement of NCG *necessarily* induces the chiral symmetry to the chiral TI. This is the resolution for (iii). The minimum

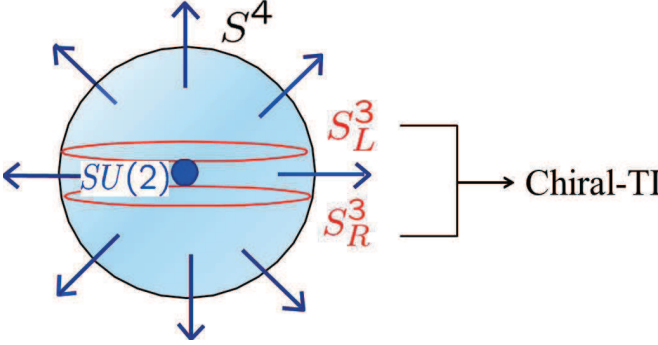


FIG. 1: Chiral TI is realized as a composite of the two three-spheres embedded in the 4D QHE, and realize the chiral symmetry in a geometrical way.

energy is achieved at $s = \pm 1/2$ for odd I not even I , and so the system becomes trivial or non-trivial depending on the parity of I . This implies a hidden \mathbb{Z}_2 structure of the chiral TI.

MANY-BODY PROBLEM

Regarding the two $s = \pm 1/2$ latitudes as “spin” degrees of freedom, we apply the Halperin’s arguments [18] to explore many-body physics of the chiral TI. The LLL basis states on the $s = +1/2$ latitude, S^3_L , are given by $\Psi_{M_L} \equiv \Psi_{l_L, m_L}^L \otimes \Psi_{l_R, m_R}^R|_{(l_L, l_R = \frac{l+1}{4}, \frac{l-1}{4})}$, and those on the $s = -1/2$ latitude S^3_R are $\Psi_{M_R} \equiv \Psi_{l_L, m_L}^L \otimes \Psi_{l_R, m_R}^R|_{(l_L, l_R = \frac{l-1}{4}, \frac{l+1}{4})}$. The total degeneracy is $D(I)(8)$. The Slater determinants on S^3_L and S^3_R are respectively constructed as $\Psi_{L-Slat} = \epsilon_{M_{L_1} M_{L_2} \dots M_{L_{N/2}}} \Psi_{M_{L_1}}(x_1) \Psi_{M_{L_2}}(x_2) \dots \Psi_{M_{L_{N/2}}}(x_{N/2})$, $\Psi_{R-Slat} = \epsilon_{M_{R_1} M_{R_2} \dots M_{R_{N/2}}} \Psi_{M_{R_1}}(x_{N/2+1}) \dots \Psi_{M_{R_{N/2}}}(x_N)$, and the Laughlin-Halperin type groundstate wavefunc-

tion is given by $\Psi_{l,l,m} \equiv \Psi_{L-Slat}^l \cdot \Psi_{R-Slat}^l \cdot \Psi_{Corr}^m$ (l : odd). Here, Ψ_{Corr}^m denotes the correlation part between S^3_L and S^3_R . We can derive the explicit form of Ψ_{Corr}^m by observing that the “spin-polarized” state $(l, l, m) = (l, l, l)$ coincides with the Laughlin wavefunction of the total particles: $\Psi_{l,l,l} = \Psi_{Lin}^{(l)} (\equiv \Psi_{T-Slat}^l)$, where $\Psi_{T-Slat} = \epsilon_{M_{T_1} M_{T_2} \dots M_{T_N}} \Psi_{M_{T_1}}(x_1) \Psi_{M_{T_2}}(x_2) \dots \Psi_{M_{T_N}}(x_N)$ with Ψ_{M_T} ($M_T = 1, 2, \dots, D(I)$) denoting the total basis states: $\Psi_{M_T} = \{\Psi_{M_L}, \Psi_{M_R}\}$. The correlation function is determined as $\Psi_{Corr} = \Psi_{T-Slat} / (\Psi_{L-Slat} \cdot \Psi_{R-Slat})$. Hence we have

$$\Psi_{l,l,m} = \Psi_{L-Slat}^{l-m} \cdot \Psi_{R-Slat}^{l-m} \cdot \Psi_{T-Slat}^m. \quad (19)$$

For $\Psi_{l,l,m}$ to be a chiral symmetric state, *i.e.* the $SU(2)$ singlet state in terms of chiral rotations, $\Psi_{l,l,m}$ has to satisfy the Fock condition and m is restricted to $m = l - 1$ [19]. Therefore, the chiral (symmetric) Laughlin-Halperin wavefunction is given by

$$\Psi_{l,l,l-1} = \Psi_{L-Slat} \cdot \Psi_{R-Slat} \cdot \Psi_{T-Slat}^{l-1} = \frac{\Psi_{Lin}^{(l)}}{\Psi_{Corr}}. \quad (20)$$

This gives a resolution for (iii). Under the scaling $I \rightarrow lI$, the total degeneracy behaves as $D(lI) \sim (lI)^2$ and so $\nu = N/D(lI) \sim 1/l^2$. To derive a precise expression of the filling factor for (l, l, m) state (19), we introduce the K -matrix [20]:

$$K = \begin{pmatrix} (l-m)^2 + m^2 & m^2 \\ m^2 & (l-m)^2 + m^2 \end{pmatrix}. \quad (21)$$

The K matrix condition is given by $K \begin{pmatrix} N_L \\ N_R \end{pmatrix} = D \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Except for $m = l$, K has the inverse and the filling factors are derived as $\begin{pmatrix} \nu_L \\ \nu_R \end{pmatrix} = \frac{1}{D} \begin{pmatrix} N_L \\ N_R \end{pmatrix} = K^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the total filling factor is

$$\nu = \nu_L + \nu_R = \frac{2}{2m^2 + (l-m)^2}. \quad (22)$$

The fractional charges are given by $(e_L^*, e_R^*) = (K^{-1}_{11}, K^{-1}_{21})$ or $(e_L^*, e_R^*) = (K^{-1}_{12}, K^{-1}_{22})$, and in either case the net fractional charge reads as $e^* = e_L^* + e_R^* = \frac{1}{2}\nu$. In particular for $(l, l, l-1)$ state, we have

$$e^* = \frac{1}{2}\nu = \frac{1}{1 + 2(l-1)^2} = 1, \frac{1}{9}, \frac{1}{33}, \frac{1}{73}, \dots \quad (23)$$

The effective field theory may also be given by the K -matrix Chern-Simons theory. Since the fundamental excitation is a string-like object coupled to 2rank tensor field, we adopt the Chern-Simons tensor field theory:

$$L_{CS} = \frac{1}{16\pi} K_{IJ} \epsilon^{\mu\nu\rho\sigma\tau} C_{\mu\nu}^I \partial_\rho C_{\sigma\tau}^J + \frac{1}{2} C_{\mu\nu}^I J^{I\mu\nu}, \quad (24)$$

which breaks the time reversal symmetry and live in 5D space-time. Since in 5D the co-dimension for two (non-overlapping) strings is 2, the string-like objects generally

obey fractional statistics. The effective field theory of A-class TI is given by the Chern-Simons tensor theory in $(4k-1)$ D space-time [12] which does not cover 5D space-time. With the effective field theories of AIII TIs, all odd dimensional the Chern-Simons tensor field theories are exhausted.

The coherent state aligned to the direction Ω_μ ($\sum_{\mu=1}^4 \Omega_\mu^2 = 1$) on S^3 satisfies the quaternionic coherent state equation:

$$\Omega_\mu \tilde{q}_\mu \chi_L = \chi_R, \quad \Omega_\mu \tilde{\bar{q}}_\mu \chi_R = \chi_L, \quad (25)$$

where $\tilde{q}_\mu = (-q_1, q_2, -q_3, 1) = (i\sigma_i^*, 1)$ and $\tilde{\bar{q}}_\mu = (q_1, -q_2, q_3, 1) = (-i\sigma_i^*, 1)$. Obviously χ_L and χ_R give

$$\chi_R^\dagger \tilde{q}_\mu \chi_L + \chi_L^\dagger \tilde{\bar{q}}_\mu \chi_R = \Omega_\mu, \quad (26)$$

and χ_L and χ_R are expressed as $\chi_L = \frac{1}{2\sqrt{1+\Omega_4}}(1 + \Omega_4 - i\Omega_i\sigma_i^*)\phi$ and $\chi_R = \frac{1}{2\sqrt{1+\Omega_4}}(1 + \Omega_4 + i\Omega_i\sigma_i^*)\phi$ with a normalized two-component spinor ϕ . The point on S^3 in the LLL is denoted as $\Omega_\mu X_\mu$. With use of the property of the quaternion $\sigma_2 \tilde{q}_\mu = \tilde{\bar{q}}_\mu \sigma_2$, it is readily shown that $\psi_\chi^{(I)} = (\chi_L^\dagger \psi_L + \chi_R^\dagger \psi_R)^I$ satisfies the coherent state equation: $\Omega_\mu X_\mu \psi_\chi^{(I)} = I \psi_\chi^{(I)}$. Creation and annihilation operators for the charged excitation generated at the point $\Omega_\mu(\chi)$ that satisfy $[\Omega_\mu X_\mu, A^\dagger(\chi)] = N A^\dagger(\chi)$, and $[A(\chi), A^\dagger(\chi)] = 1$, are constructed as

$$\begin{aligned} A^\dagger(\chi) &= \prod_{i=1}^N (A_L^\dagger(\chi_L)_i + A_R^\dagger(\chi_R)_i), \\ A(\chi) &= \prod_{i=1}^N (A_L(\chi_L)_i + A_R(\chi_R)_i), \end{aligned} \quad (27)$$

where $A_L^\dagger(\chi_L)_i \equiv i\chi_L^t \sigma_2 \psi_L(i)$, $A_L(\chi_L)_i \equiv i\chi_L^\dagger \sigma_2 \frac{\partial}{\partial \psi_L(i)}$ and similar expressions for R . The chiral operators $A_L(\chi_L)_i$ and $A_R(\chi_R)_i$ satisfy $[A_L(\chi_L)_i, A_L^\dagger(\chi_L)_j] = [A_R(\chi_R)_i, A_R^\dagger(\chi_R)_j] = \delta_{ij}$, and $[A_L(\chi_L)_i, A_L(\chi_L)_j] = [A_L(\chi_L)_i, A_R^\dagger(\chi_R)_j] = [A_R(\chi_R)_i, A_R(\chi_R)_j] = 0$. Due to the relation $\chi_L^\dagger \chi_L = \chi_R^\dagger \chi_R = 1/2$, either of $A_L^\dagger(\chi_L)$ and $A_R^\dagger(\chi_R)$ cannot be zero. Since $\psi = (\psi_L, \psi_R)^t$ is a $SO(4)$ Dirac spinor that carries the $SU(2)_L \oplus SU(2)_R$ bispin $(j_L, j_R) = (1/2, 0) \oplus (0, 1/2)$, $A(\chi)$ and $A(\chi)^\dagger$ denote non-chiral operators for charged excitation with left and right chiralities.

THE COLOR HALL EFFECT

While the chiral TI shares similar properties with QHE such as time-reversal breaking and \mathbb{Z} classification of topological number, \mathbb{Z}_2 structure is also incorporated in the chiral TI due to the chiral symmetry, just like the time-reversal symmetry of the QSHE. Therefore, the chiral TI is expected to accommodate a dual property of the

QHE and QSHE. Indeed, the dual property will be manifest in the transport phenomena.

Name the two $SU(2)$ color indices L and R and the three colors of $SU(2)$ gauge fields $a = 1, 2, 3$. Since the color gauge fields are independently coupled to the corresponding color currents, the Hall effect is given by

$$J_i^a = \sigma \epsilon_{ijk} E_j^a B_k^a. \quad (\text{no sum for } a) \quad (28)$$

Without loss of generality, we focus on $a = 3$ in which $J_i^3 = J_i^L - J_i^R$. If there only exist either of L or R -color particles, the Hall effect will be given by $J_i^L = \sigma_L \epsilon_{ijk} E_j B_k$ and $J_i^R = -\sigma_R \epsilon_{ijk} E_j B_k$. The L and R -color currents flow in the mutually opposite direction, similar to the spin Hall effect where flows of up and down spin currents are opposite. However, the color Hall effect does not respect the time reversal symmetry, since the L and R are just labels and are not flipped under the time reversal transformation unlike physical spin of the spin Hall effect. The time reversal transformation just reverses the direction of the L and R -color currents as in the case of the ordinary Hall effect. The quantized version of the color Hall effect can similarly be understood. L and R -color currents respectively contribute to the quantized Hall conductivity as $\sigma_L = \frac{e^2}{2\pi\hbar} \nu_L$, $\sigma_R = -\frac{e^2}{2\pi\hbar} \nu_R$, where e denotes the color coupling constant. The total and different conductances are obtained as

$$\begin{aligned} \sigma &\equiv \sigma_L + \sigma_R = (\nu_L - \nu_R) \frac{e^2}{2\pi\hbar}, \\ \Delta\sigma &\equiv \sigma_L - \sigma_R = (\nu_L + \nu_R) \frac{e^2}{2\pi\hbar}. \end{aligned} \quad (29)$$

For the $(l, l, l-1)$ chiral TI, we have a non-chiral version of the Hall effect: $\sigma = 0$, $\Delta\sigma = \frac{1}{1+2(l-1)^2} \frac{e^2}{\pi\hbar}$, which reduces to the QSH conductance $\Delta\sigma_{\text{QSH}} = \frac{e^2}{\pi\hbar}$ [21] for $l = 1$.

SUMMARY AND DISCUSSIONS

To summarize, we explored one-particle and many-body physics of the chiral TI with Landau level based on the Nambu 3-algebraic geometry. The chiral TI is a natural 3D generalization of the Haldane's 2D QHE and 3D "reduction" of the Zhang and Hu's 4D QHE. We elucidated the former controversial problems by exploiting the mathematics and physics of the chiral TI. In particular, we clarified that Nambu 3-algebraic geometry is essential for the existence of the chiral symmetry of the chiral TI. Interestingly, the chiral TI exhibits a dual property of QHE and QSHE due to the hidden \mathbb{Z}_2 structure of the chiral symmetry.

Recently, Li and Wu constructed 3D AII TI model with Landau level [22]. Though their model also heavily utilized quaternionic structure, the model respects the time-reversal symmetry and hence describes different physics.

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- [1] Titus Neupert, Luiz Santos, Shinsei Ryu, Claudio Chamon, Christopher Mudry, “Noncommutative geometry for three-dimensional topological insulators”, Phys. Rev. B 86 (2012) 035125; arXiv:1202.5188.
 - [2] B. Estienne, N. Regnault, B. A. Bernevig, “D-Algebra Structure of Topological Insulators”, Phys. Rev. B 86 (2012) 241104(R); arXiv:1202.5543.
 - [3] Yoichiro Nambu, “Generalized Hamiltonian Dynamics”, Phys.Rev.D7 (1973) 2405-2412.
 - [4] Thomas Curtright, Cosmas Zachos, “Classical and Quantum Nambu Mechanics”, Phys.Rev.D68 (2003) 085001; hep-th/0212267.
 - [5] Joshua DeBellis, Christian Saemann, Richard J. Szabo, “Quantized Nambu-Poisson Manifolds and n -Lie Algebras”, J.Math.Phys.51 (2010) 122303; arXiv:1001.3275.
 - [6] Anirban Basu, Jeffrey A. Harvey, “The $M2$ - $M5$ Brane System and a Generalized Nahm’s Equation”, Nucl.Phys. B713 (2005) 136-150; hep-th/0412310.
 - [7] Jonathan Bagger, Neil Lambert, “Modeling Multiple $M2$ ’s”, Phys.Rev.D75 (2007) 045020; hep-th/0611108.
 - [8] Andreas Gustavsson, “Algebraic structures on parallel $M2$ -branes”, Nucl.Phys.B811 (2009) 66-76; arXiv:0709.1260.
 - [9] Jonathan Bagger, Neil Lambert, “Gauge Symmetry and Supersymmetry of Multiple $M2$ -Branes”, Phys.Rev.D77 (2008) 065008; arXiv:0711.0955.
 - [10] Shinsei Ryu, Andreas Schnyder, Akira Furusaki, Andreas Ludwig “Topological insulators and superconductors: ten-fold way and dimensional hierarchy”, New J. Phys. 12 (2010) 065010; arXiv:0912.2157.
 - [11] Pavan Hosur, Shinsei Ryu, Ashvin Vishwanath, “Chiral Topological Insulators, Superconductors and other competing orders in three dimensions”, Phys. Rev. B, 81 (2010) 045120; arXiv:0908.2691.
 - [12] Kazuki Hasebe, “Higher dimensional quantum Hall effect as A-class topological insulator”, arXiv:1403.5066.
 - [13] V.P. Nair, S. Randjbar-Daemi, “Quantum Hall effect on S^3 , edge states and fuzzy S^3/\mathbf{Z}_2 ”, Nucl.Phys. B679 (2004) 447-463; hep-th/0309212.
 - [14] F.D.M. Haldane, “Fractional quantization of the Hall effect: a hierarchy of incompressible quantum fluid states”, Phys. Rev. Lett. 51 (1983) 605-608.
 - [15] S.-C. Zhang, J.-P. Hu, “A four-dimensional generalization of the quantum Hall effect”, Science 294 (2001), no. 5543, 823-828; cond-mat/0110572.
 - [16] Sanjaye Ramgoolam, “Higher dimensional geometries related to fuzzy odd-dimensional spheres”, JHEP 0210 (2002) 064; hep-th/0207111.
 - [17] M. M. Sheikh-Jabbari, M. Torabian, “Classification of All $1/2$ BPS Solutions of the Tiny Graviton Matrix Theory”, JHEP 0504 (2005) 001; hep-th/0501001.
 - [18] B. I. Halperin, “Theory of the quantized Hall conductance”, Helv. Phys. Acta 56 (1983) 75.
 - [19] D. Yoshioka, A. H. MacDonald, S. M. Girvin, “Connection between spin-singlet and hierarchical wave functions in the fractional quantum Hall effect”, Phys. Rev. B 38 (1988) 3636-3639.
 - [20] X.G. Wen, A. Zee, “Classification of Abelian quantum Hall states and matrix formulation of topological fluids”, Phys. Rev. B 46 (1992) 2290-2301.
 - [21] B.A. Bernevig and S.C. Zhang, “Quantum Spin Hall Effect”, Phys. Rev. Lett. 96, 106802 (2006); cond-mat/0504147.
 - [22] Yi Li, Congjun Wu, “High-Dimensional Topological Insulators with Quaternionic Analytic Landau Levels”, Phys. Rev. Lett. 110 (2013) 216802; arXiv:1103.5422.
 - [23] Even though the chiral TIs are refereed to “chiral”, they respect chiral symmetry and hence they are non-chiral in the language of high energy physics.
 - [24] We adopt $I/2$ and n instead of J and q in [13]. s is related to the extra parameter μ in [13] by $s = \mu - \frac{1}{2}I$.
 - [25] $\bar{}$ represents the quaternionic conjugation.
 - [26] $q_\mu \bar{q}_\nu - q_\nu \bar{q}_\mu = -2\bar{\eta}_{\mu\nu}^i q_i$ and $\bar{q}_\mu q_\nu - \bar{q}_\nu q_\mu = -2\eta_{\mu\nu}^i q_i$